$$(NLS): \begin{cases} i\partial_t u + \Delta u = |u|^2 u, & (t,x) \in \mathbb{R} \times \mathbb{T}^2 \\ u|_{t=0} = u_0 \in H^2(\mathbb{T}^2) \end{cases}$$

we have already shown that (NLS) possesses a global solution $u(t) \in C(\mathbb{R}; H^2(\mathbb{T}^2))$ that growth at most $\exp(\exp(ct))$.

In this lecture, we would like to understand more precise growth rate as $t \to +\infty$.

why this is important ?

Recall: (NLS) is a Hamiltonian system with conserved mass

$$M[u(t)] = \int_{\mathbb{T}^2} |u(t,x)|^2 dx$$

and energy

$$H[u(t)] = \int_{\mathbb{T}^2} \frac{1}{2}|\nabla u|^2 + \frac{1}{4}|u|^4 dx$$

Consequently, the L^2-norm and H^1-norm remains bounded (for smooth solution)

However, for high dimension $d \geq 2$, there is no (at least not known) other conservation laws for (NLS), so it is not clear whether other Sobolev norms will remains bounded.

In fact, people believe that (NLS) exhibits weak-turbulence phenomenon, i.e. a cascade of energy from large scales (low-frequencies) to small scales (high frequencies).

If such phenomenon happens, then the H^s-norm $(s>1)$ of such solution will growth, as $\|u(t)\|_{H^s}^2 = \sum_{k \in \mathbb{Z}^2} |\hat{u}(t,k)|^2 (|k|^{2s}+1)$.

if some portion of energy transfer from small k to large k, then $\|u\|_{H^s}^2$ will get larger,

Mathematically, we have the conjecture of Bourgain:

$\boxed{\text{Conjecture:}}$ Does (NLS) possesses a (smooth) solution $u(t)$

such that $\limsup_{t \to +\infty} \| u(t) \|_{H^2(\mathbb{T}^2)} = +\infty$

☆ only partial progress had been made by [Colliander - Keel - Staffilani - Takaoka - Tao, 2010]

we could adress a simple question on the opposite side.

Q: How fast high-order Sobolev norms can grow along the flow associated with Hamiltonian PDEs.

results: Bourgain 1993. 1996. 1999.

Colliander et al. 2012.

Delort 2014

So we are going to prove the following polynomial bound.

$\boxed{\text{Theorem}}$ (polynomial growth)

$\exists A>0$. For all global solution $u(t)\in C(\mathbb{R}; H^2(\mathbb{T}^2))$.

we have $\|u(t)\|_{H^2(\mathbb{T}^2)} \leq C_\sigma(\|u(0)\|_{H^2(\mathbb{T}^2)})\, t^A$. $\forall t\geq 1$.

$\boxed{\text{Remark:}}$ The proof relies on a clear energy estimate (following Planchon - Tzvetkov - Visciglia, 2017') and a discrete Fourier restriction type estimate of J. Bourgain.

$\boxed{\text{Idea}}$: we do estimate for Δu in the previous proof.

$$i\partial_t u + \Delta u = |u|^2 u$$

look at L^2- norm of $\partial_t u$.

$$\|\partial_t u\|_{L^2} \approx \|\Delta u\|_{L^2} + \underbrace{\||u|^2 u\|_{L^2}}_{\leq \|u\|^3_{L^6(\mathbb{T}^2)} \leq \underbrace{\|u\|^3_{H^1(\mathbb{T}^2)}}_{\text{lower regularity term}}}$$

$\Rightarrow$ Try to do estimate for $\|\partial_t u\|_{L^2(\mathbb{T}^2)}$

Pf:
$$\begin{aligned}
\frac{d}{dt}\|\partial_t u\|^2_{L^2(\mathbb{T}^2)} &= \frac{d}{dt}\int_{\mathbb{T}^2}|\partial_t u|^2\,dx \\
&= 2\,\mathrm{Re}\int_{\mathbb{T}^2}\partial_t\partial_t u\,\partial_t\bar{u}\,dx \\
&= 2\,\mathrm{Re}\int_{\mathbb{T}^2}\partial_t(i\Delta u - i|u|^2u)\,\partial_t\bar{u}\,dx \\
&= 2\,\mathrm{Re}\left(i\int_{\mathbb{T}^2}\Delta\partial_t u\,\partial_t\bar{u}\,dx\right) - 2\,\mathrm{Re}\left(i\int_{\mathbb{T}^2}\partial_t(|u|^2u)\,\partial_t\bar{u}\,dx\right) \\
&= \cancel{-2\,\mathrm{Re}\left(i\int|\nabla\partial_t u|^2\,dx\right)} - \cancel{2\,\mathrm{Re}\left(i\int_{\mathbb{T}^2}\partial_t u\,\partial_t\bar{u}\,|u|^2\,dx\right)} \\
&\quad - 2\,\mathrm{Re}\left(i\int_{\mathbb{T}^2}\partial_t(|u|^2)\,u\,\partial_t\bar{u}\,dx\right) \\
&= -2\,\mathrm{Re}\, i\int_{\mathbb{T}^2}\partial_t(|u|^2)\,u\,\partial_t\bar{u}\,dx \\
&= -2\,\mathrm{Re}\, i\int_{\mathbb{T}^2}\partial_t(|u|^2)\,u\,(-i\Delta\bar{u} + i|u|^2\bar{u})\,dx \\
&= \underbrace{-2\,\mathrm{Re}\int_{\mathbb{T}^2}\partial_t(|u|^2)\,u\Delta\bar{u}\,dx}_{\mathrm{I}} + \underbrace{2\,\mathrm{Re}\int_{\mathbb{T}^2}\partial_t(|u|^2)\,|u|^4\,dx}_{\mathrm{II}}
\end{aligned}$$

$$\Delta(|u|^2) = \Delta(u\cdot\bar{u}) = \nabla(\nabla u\cdot\bar{u} + u\cdot\nabla\bar{u})$$
$$= \Delta u\cdot\bar{u} + |\nabla u|^2 + \nabla u\cdot\nabla\bar{u} + u\,\Delta\bar{u}$$
$$= \Delta u\cdot\bar{u} + 2|\nabla u|^2 + u\cdot\Delta\bar{u}$$
$$= 2\mathrm{Re}(u\,\Delta\bar{u}) + 2|\nabla u|^2$$

$$2\mathrm{Re}(u\,\Delta\bar{u}) = u\cdot\Delta\bar{u} + \Delta u\cdot\bar{u}$$

So $\mathrm{I} = -2\mathrm{Re}\int_{\mathbb{T}^2}\partial_t(|u|^2)\,u\cdot\Delta\bar{u}\,dx = -\int_{\mathbb{T}^2}\partial_t(|u|^2)\left[\Delta(|u|^2) - 2|\nabla u|^2\right]dx$

$$= \int_{\mathbb{T}^2}\partial_t(\nabla|u|^2)(\nabla\cdot|u|^2) + 2\int_{\mathbb{T}^2}\partial_t(|u|^2)|\nabla u|^2 dx$$

$$= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}|\nabla|u|^2|^2 + 2\int_{\mathbb{T}^2}\partial_t(|u|^2)|\nabla u|^2 dx$$

on the other hand.

$$\mathrm{II} = 2\mathrm{Re}\int_{\mathbb{T}^2}\partial_t(|u|^2)\,|u|^4\,dx$$
$$= \frac{3}{2}\frac{d}{dt}\int_{\mathbb{T}^2}|u|^6\,dx$$

$$\frac{d}{dt}|u|^6 = \frac{d}{dt}(|u|^2|u|^4)$$
$$= \frac{d}{dt}(|u|^2)|u|^4 + |u|^2\frac{d}{dt}(|u|^2|u|^2)$$
$$= \partial_t(|u|^2)|u|^4 + 2|u|^4\partial_t(|u|^2)$$
$$= 3|u|^4\partial_t(|u|^2)$$

So we have

$$\frac{d}{dt}\|\partial_t u\|^2_{L^2(\mathbb{T}^2)} = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}|\nabla|u|^2|^2 + 2\int_{\mathbb{T}^2}\partial_t(|u|^2)|\nabla u|^2 dx + \frac{3}{2}\frac{d}{dt}\int_{\mathbb{T}^2}|u|^6\,dx$$

Idea: construct a modified energy:

Define $\mathcal{E}_m[u(t)] = \|\partial_t u\|^2_{L^2(\mathbb{T}^2)} \underbrace{- \frac{3}{2}\|u(t)\|^6_{L^6(\mathbb{T}^2)} - \frac{1}{2}\|\nabla|u|^2\|^2_{L^2(\mathbb{T}^2)}}_{\text{lower regularity term}}$

because $\|u\|_{L^6} \le \|u\|_{H^1(\mathbb{T}^2)}$

$\|\nabla|u|^2\|_{L^2} \approx \|\nabla u\cdot u\|_{L^2}$

$\le \|u\|_{H^1}\|u\|_{L^2}$.

we then have $\frac{d}{dt}\mathcal{E}_m[u(t)] = 2\int_{\mathbb{T}^2}|\nabla u|^2\partial_t(|u|^2)\,dx$

If we look at $\partial_t(|u|^2)$.

$$\partial_t(|u|^2) \approx 2\partial_t u\,|u| \simeq 2(i\Delta u - i|u|^2 u)|u|$$

So $$|\nabla u|^2 \partial_t(|u|^2) \approx \underbrace{|\nabla u|^2}_{L^4\ L^4} \underbrace{\Delta u}_{L^2} \cdot \underbrace{|u|}_{L^\infty} - |u|^4$$

Recall: $0 < s < \frac{d}{2}$. $H^s \subseteq L^p$. $p = \frac{2d}{d-2s}$ $\rightsquigarrow$ $H^{1/2} \subseteq L^4(\mathbb{T}^2)$

$$\|\nabla u\|_{L^4}^2 \leq \|u\|_{H^{3/2}(\mathbb{T}^2)}^2 \underset{\substack{\uparrow \\ \text{interpolation inequality}}}{\leq} \|u\|_{H^2}^{3/2} \|u\|_{L^2}^{\frac{1}{2}}$$

$$\|u\|_{H^{3/2}(\mathbb{T}^2)} \leq \|u\|_{H^2(\mathbb{T}^2)}^{3/4} \|u\|_{L^2(\mathbb{T}^2)}^{1/4}$$

$$\Rightarrow \underbrace{|\nabla u|^2}_{\|u\|_{H^2}^{3/2}} \underbrace{\Delta u}_{\|u\|_{H^2}} \underbrace{|u|}_{L^\infty} \rightsquigarrow \|u\|_{H^2}^{5/2}, \quad \text{not good}$$

Idea: do time estimate to gain some integrability.

$$\int_0^t \frac{d}{ds} \mathcal{E}_m[u(s)]\,ds \qquad \forall\, 0 < t \leq T$$

$$= \mathcal{E}_m[u(t)] - \mathcal{E}_m[u(0)]$$

$$\leq \int_0^t \|\nabla u\|_{L_x^4}^2 \|u\|_{H_x^2} \|u\|_{L_x^\infty}\,ds$$

$$\leq t^{1/2} \|\nabla u\|_{L_s^4([0,T];L_x^4)}^2 \|u\|_{L_s^\infty([0,T), H_x^2)} \|u\|_{L_s^\infty([0,T]; L_x^\infty)}$$

Lemma (Bourgain 1993')

Claim: $$\|\nabla u(t)\|_{L_t^4([0,T]; L_x^4(\mathbb{T}^2))} \lesssim (T^A + 1)\Big(\|\nabla(|u|^2 u)\|_{L_t^\infty H^\varepsilon} + \|u_0\|_{H_x^2}\Big)$$

provided that $i\partial_t u + \Delta u = |u|^2 u$

So $$\|\nabla u(t)\|_{L_t^4([0,T]; L_x^4(\mathbb{T}^2))}^2 \lesssim (T^A + 1)^2 \Big[\|u\|_{L_t^\infty L_x^\infty}^4 \|\nabla u\|_{L_t^\infty H^\varepsilon}^2 + \underline{\|u_0\|_{H_x^2}^2}\Big]$$

$$\Rightarrow \mathcal{E}_m[u(t)] - \mathcal{E}_m[u(0)]$$

$$\leq t^{1/2}\Big[(t^A + 1)^2 \big[\|u\|_{L_s^\infty L_x^\infty}^4 \|\nabla u\|_{L_s^\infty H^\varepsilon}^2 + \|u_0\|_{H_x^2}^2\big]\Big] \|u\|_{L_s^\infty([0,T), H_x^2)} \|u\|_{L_s^\infty([0,T]; L_x^\infty)}$$

$$\lesssim t^{1/2}(t^{\tilde{A}} + 1)\|u\|_{L_s^\infty([0,T]; L_x^\infty)}^5 \|u\|_{L_s^\infty([0,T]; H^{1+\varepsilon})}^2 \|u\|_{L_s^\infty([0,T]; H^2)}$$

Recall Brezis-Gallouet :

$$\|u\|_{L^\infty_x} \lesssim \|u\|_{H^1_x} \left[1 + \log^{1/2} \left(1 + \frac{\|u\|_{H^2_x}}{\|u\|_{H^1_x}} \right) \right]$$

$$\Rightarrow \quad \|u\|^5_{L^\infty_x} \lesssim \|u\|^5_{H^1_x} \left[1 + \log^5 \left(1 + \frac{\|u\|_{H^2_x}}{\|u\|_{H^1_x}} \right) \right]$$

$$\lesssim \|u\|^5_{H^1_x} \left(1 + \|u\|^{\varepsilon'}_{H^2_x} \right)$$

By $\log f \leq f^{\varepsilon}$.

$$\Rightarrow \quad \mathcal{E}_m[u(t)] \leq \mathcal{E}_m[u(0)] + T^{\tilde{\tilde{A}}} \|u\|^5_{L^\infty_s(0,T;H^1_x)} \|u\|^2_{L^\infty_s([0,T];H^{1+\varepsilon})} \|u\|^{1+\varepsilon'}_{L^\infty_s([0,T];H^2)}$$

$$\Rightarrow \|u(T)\|^2_{H^2_x(\mathbb{T}^2)} \leq \sup_{t\in[0,T]} \|u(t)\|^2_{H^2(\mathbb{T}^2)}$$

$$\lesssim \sup_{t\in[0,T]} \mathcal{E}_m[u(t)]$$

$$\leq C\|u(0)\|^2_{H^2} + T^{\tilde{\tilde{A}}} \|u\|^5_{L^\infty_s(0,T;H^1_x)} \|u\|^2_{L^\infty_s([0,T];H^{1+\varepsilon})} \|u\|^{1+\varepsilon'}_{L^\infty_s([0,T];H^2)}$$

Interpolation

$$\boxed{\|f\|_{H^{1+\varepsilon}} \leq \|f\|^{1-\varepsilon}_{H^1} \|f\|^{\varepsilon}_{H^2}} \quad \leq C\|u(0)\|^2_{H^2} + T^{\tilde{\tilde{A}}} \|u\|^{5+2-\varepsilon}_{L^\infty_s(0,T;H^1_x)} \underbrace{\|u\|^{1+\varepsilon'+\varepsilon}_{L^\infty_s([0,T];H^2)}}$$

By inequality : $a^2 \leq b + cT^{\tilde{A}} a^{1+\sigma} \Rightarrow a \lesssim c' T^{\tilde{A}'}$

So $\|u(T)\|_{H^2_x} \leq C_\sigma(\|u(0)\|_{H^2}) \, T^{A'}$

$\square$